

Disorder Induced Limited Path Percolation

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Abstract. - We introduce a model of percolation induced by disorder, where an initially homogeneous network with links of equal weight is disordered by the introduction of heterogeneous weights for the links. We consider a pair of nodes i and j to be mutually reachable when the ratio α_{ij} of length of the optimal path between them before and after the introduction of disorder does not increase beyond a tolerance ratio τ . These conditions reflect practical limitations of reachability better than the usual percolation model, which entirely disregards path length when defining connectivity and, therefore, communication. We find that this model leads to a first order phase transition in both 2-dimensional lattices and in Erdős-Rényi networks, and in the case of the latter, the size of the discontinuity implies that the transition is effectively catastrophic, with almost all system pairs undergoing the change from reachable to unreachable. Using the theory of optimal path lengths under disorder, we are able to predict the percolation threshold. For real networks subject to changes while in operation, this model should perform better in predicting functional limits than current percolation models.

Percolation theory is the most well-established approaches to study system connectivity, and how this connectivity becomes compromised with local system failure [1]. Systems usually refer to connected structures such as lattices or random networks [2], and failures to the removal of nodes or links. Percolation predicts failure thresholds and size of the connected parts of the network after those failures. Practical domains of applications include epidemiology [3–5], communication networks like the Internet [6], and propagation of information in social networks [7].

Recently [8], it was pointed out that there are many contexts in which structural connectivity may not be sufficient for a network to maintain its functionality when faced with failures. Network nodes may need to be more than connected; they may need to be reachable from one another, which is practical definition. In regular percolation, node j is reachable from node i if there is a path of consecutive links from i to j , i.e., if the nodes are structurally connected after any failures. However, in real networks, two nodes initially reachable from one another through a structural path can become unreachable after link failures even if

there is a new structural path connecting them, but which is in some way unsatisfactory. This reasoning lead to the introduction of Limited Path Percolation (LPP), in which reachability between nodes i and j after the onset of failures is a relative statement: if the path length between nodes before (ℓ_{ij}) and after (ℓ'_{ij}) failures is such that $\ell'_{ij}/\ell_{ij} \leq \tau$, then node j is reachable from i (and vice versa), where τ is an externally imposed tolerance that reflects properties of the functional aspect of the network (regular percolation corresponds to $\tau \rightarrow \infty$). For a large class of networks, Ref. [8] shows that a new phase transition appears, where networks are now more fragil due to the additional length constraint.

The approach introduced in Ref. [8] was restricted to the removal (absolute failure) of some links, while other links remained unperturbed. However, in many real-world examples, links usually do not fail entirely, but can instead become more difficult to use such as in road networks affected by weather. One can then imagine a network to begin with no performance degradation (equally useful links forming an “ordered” network), but progressively develop some form of disorder in which each link becomes costly to use (with cost or weight w). In other words, the network acquires a distribution of link weights $P_a(w)$ where a is the disorder parameter [9]. Once disorder has set in, new “optimal” paths (paths of least total weight) between each pair of nodes must be found, and a new *disorder induced Limited Path Percolation* can be defined on the basis of the path length change between the original unweighted network and the subsequent weighted network, reflecting a scenario more akin to many real networks.

In this article, we study disorder induced LPP, and find that the model leads to a first order phase transition for both lattices and Erdős-Rényi networks, indicating a catastrophic failure of the system when the disorder exceeds the tolerance limit of the network. For Erdős-Rényi networks, the failure appears to be catastrophic, as our numerical results indicate that the phase transition occurs between an almost fully connected network to a network of fractional size zero in the thermodynamic limit. Through the use of theoretical results that explain the behavior of optimal paths, we are able to predict the location of the LPP transition in both network structures as a function of the disorder, and structural properties of the system. Our model also displays universal features with respect to the disorder, as demonstrated by our numerical results.

Model and Methods. –

Formulation of Limited Path Percolation induced by disorder. To be concrete, LPP in Ref. [8], where link failure corresponds to link removal, is formulated in the following way: if a pair of nodes i and j is connected through a shortest path of length ℓ_{ij} (number of links) at $p = 1$ (no links removed), and of length $\ell'_{ij}(p)$ at $p < 1$ (fraction $1 - p$ of links removed), i and j are considered reachable if $\ell'_{ij}(p) \leq \tau \ell_{ij}$, where τ is the tolerance factor which lies on the range between 1 and ∞ [8]. The number of reachable pairs, S , is measured through

$$S = \sum_{i \neq o} \theta(\tau \ell_{oi} - \ell'_{oi}) \quad (1)$$

where o is an “origin” node chosen to minimize its impact on S (see below), and θ is the Heaveside step function $\theta(x) = 1$ if $x \geq 0$, and 0 otherwise. S depends on both p and τ , and the LPP phase transition occurs for the combination of these parameters at the threshold of the relation $S \sim N$. For fixed τ , there is a threshold $p = \tilde{p}_c(\tau)$ at which $S \sim N$ is achieved. Alternatively, given $p \geq p_c$ [10], there is a $\tau = \tau_c(p)$ such that $S \sim N$. When $\tau \rightarrow \infty$, $\tilde{p}_c(\tau \rightarrow \infty) = p_c$.

For disorder induced LPP, the set up is similar, but instead of considering each link to be kept with probability p , we change each link weight from 1 to w drawn from a random uncorrelated distribution $P_a(w)$ where a is the disorder parameter (defined below) [11]. The path lengths change from ℓ_{ij} to $\ell'_{ij}(a)$ where now the later corresponds to the length of the optimal path between i and j . Reachability is then defined as in Ref. [8]: i and j are

considered reachable if $\ell'_{ij}(a) \leq \tau \ell_{ij}$. The number of node pairs S that remain reachable is a function of a , τ , and N . We search for the LPP threshold by imposing $S \sim N$. If a threshold exists for a fixed tolerance τ , there is a critical disorder $a = a_c(\tau)$ or, vice versa, a critical tolerance $\tau = \tau_c(a)$ that depends on the disorder parameter a . Another way to phrase this is to say that τ and a are control parameters in disorder induced LPP, as τ and p are in regular LPP.

To consider a tolerance to the path length increase without considering an associated tolerance to the path weight increase may at first seem arbitrary, but in fact is well justified in that the overall weight of a path is asymptotically proportional to its length under the conditions of disorder we study here. Hence, choosing path length tolerance does not affect the qualitative nature of our results, and one choice of tolerance can be mapped onto the other. In concrete terms, if we imagined disorder corresponding to something like time of travel through a link, then total average travel time scales linearly with the trip distance.

The general algorithm used to measure disorder induced LPP is the following. First, we select the ensemble of networks \mathcal{G} of interest. In this article we focus on 2-dimensional square lattices and Erdős-Rényi (ER) graphs [12]. Square lattices have no randomness, of course, but can be formally viewed as an ensemble with one single realization. For each network realization $G \in \mathcal{G}$, in which all links have weight 1, we determine the path length ℓ_{oi} between nodes o and i for all $i \neq o$. On the same network realization G (the same nodes and links), disorder is introduced by changing the weight of each link from 1 to w_{ij} . Subsequently, the optimal paths between o and all other nodes i of G are determined, and their lengths ℓ'_{oi} recorded. S is calculated by using Eq. (1). To determine path lengths, we use the Dijkstra algorithm [13].

Disorder. We consider disorder distributions $P_a(w)$ characterized by a single disorder parameter which, for convenience, we label as a . Generally, we would expect that disorder induced LPP changes as a function of the form of $P_a(w)$. However, recent work [14] shows that large classes of disorder distributions are essentially equivalent in the optimal path problem, provided a certain characteristic length scale $\xi(a)$ associated with $P_a(w)$ is conserved (see below). In practical terms, this means that disorder distributions of different functional forms but with equal $\xi(a)$ lead to optimal path distributions that scale in the same way [14].

This result allows us to choose a distribution that is convenient and well understood. Thus, we use

$$P_a(w) = (aw)^{-1}, \quad [w \in [1, e^a]] \quad (2)$$

for which a large amount of research has been conducted in the context of the optimal path problem [15–19]. Determining ξ for a given distribution is addressed in [14], and we return to this in the discussion of results.

Results and Discussion. – To characterize LPP effectively, we first measure $\Theta(S|\tau, a, N)$, the distribution of sizes of the cluster containing o , for networks of size N with tolerance τ over disorder and network realizations. Previously [8], the LPP transition was found by calculating $\langle S \rangle = \sum_S S \Theta(S)$ and determining the parameter values at the threshold of $\langle S \rangle \sim N$. Here we develop a more systematic approach, in which we look at the entire distribution $\Theta(S|\tau, a, N)$ in order to explore whether disorder induced LPP exhibits a phase transition, and if it does, what is the order. To study the thermodynamic limit, we find it useful to analyze the fractional mass $\sigma \equiv S/N$, and hence $\Theta(\sigma|\tau, a, N)$ (or $\Theta(\sigma|\tau, a, L)$ for lattices).

On a square lattice of equal sides L and $N = (L+1)^2$ nodes, we measure $\Theta(\sigma|\tau, a, L)$ from node o located at the center of the lattice ($x_o = [(L+1)/2]$, $y_o = [(L+1)/2]$) with $[.]$ indicating the next lowest integer of the argument. In Fig. 1(a), we show $\Theta(\sigma|\tau, a = 10, L)$ with several L and τ . The first interesting feature is the narrow shape of the distribution, indicating the presence of a characteristic mass S for given τ and L . This suggests focusing

on the most probable value of σ , labeled $\sigma^*(\tau, a, N)$, i.e. $\Theta(\sigma^*|\tau, a, N) > \Theta(\sigma|\tau, a, N)$ for all $\sigma \neq \sigma^*$. Also, we observe that for small τ , as the system size increases, σ^* systematically decreases, but in contrast, for large τ , σ^* increases. Between these two cases, we find a τ for which $\sigma^*(\tau)$ remains virtually constant. For such τ , labeled $\tau_c^{(\text{Latt})}$, $S \sim N$ since $\sigma^*(\tau_c) = \text{const.}$, signaling the appearance of a phase transition. From Fig. 1(a) we observe that $\sigma^*(\tau_c^{(\text{Latt})})$ is close to 0.20, suggesting a first order transition.

A systematic study of $\sigma^*(\tau, a, N)$ can be carried out with the purpose of understanding in more detail the phase diagram of the model. In Fig. 1(b) we present σ^* for $a = 10$ and a range of L and τ . The value of σ^* was estimated from $\Theta(\sigma|\tau, a, L)$ by finding an appropriate cubic fit for the peak of the distribution and calculating the location of its maximum. Three main regimes can be observed for τ above, close to, and below $\tau_c^{(\text{Latt})}$. The determination of $\tau_c^{(\text{Latt})}$ as a function of a from simulations is done by inspection and requires exploring a range of τ with considerable precision ($\delta\tau \sim 10^{-2}$ or even 10^{-3} , becoming more sensitive for large a) around a certain region, in plots such as Fig. 1(b). For small τ (say, close to 1 and well below $\tau_c^{(\text{Latt})}$), it is interesting to see that as the system size increases, $\sigma^* \sim L^{-2}$ indicating that the fraction of the system that is reachable is smaller than any power of N . The situation for $\tau \lesssim \tau_c^{(\text{Latt})}$ is not as clear: in LPP due to link removal [8] there is a regime of power law sizes of S , whereas here such regime is not evident for lattices but seems to be present for ER networks (see below); our current theory does not shed light on the matter. For $\tau > \tau_c^{(\text{Latt})}$, we find a progressive increase of σ^* with respect to L , with a saturation value that depends on τ ; σ^* gradually approaches 1 as $\tau \rightarrow \infty$.

To measure $\Theta(\sigma|\tau, a, N)$ in ER networks, we sample over network realizations of $G \in \mathcal{G}$, and for each G choose an origin o at random. In Fig. 1(c) we present the relevant simulation results. The qualitative features of $\Theta(\sigma|\tau, a, N)$ for lattices are also present in ER networks, including the existence of a critical τ , labeled $\tau_c^{(\text{ER})}$. In contrast to lattices, $\sigma^*(\tau_c^{(\text{ER})})$ is close to 1, indicating a very dramatic LPP transition, in which a slight change of τ around $\tau_c^{(\text{ER})}$ leads to a transition between almost entirely reachable to entirely unreachable global network states. We also observe that for $\tau < \tau_c^{(\text{ER})}$, there seems to be a power-law decaying relation between σ^* and N , consistent with a fractal size object below the threshold, with the decay exponent τ -dependent.

To analyze the problem further, we define the quantity $\alpha_{ij} = \ell'_{ij}/\ell_{ij}$, called the length factor, for each node pair ij [20], which measures the fractional increase of the path between i and j , and explore the distribution $\Phi(\alpha)$ and its cumulative $F(\alpha) = \int^\alpha \Phi(\alpha') d\alpha'$ over realizations of $P_a(w)$ (and \mathcal{G} for ER networks). Note that the LPP reachability condition is $\alpha_{ij} \leq \tau$.

Figure 2(a) shows the measurement of $F(\alpha|a, L)$ in lattices, and in the inset the distribution $\Phi(\alpha|a, L)$, which is a well concentrated function around its maximum $\alpha = \alpha_c$. The cumulative $F(\alpha)$ increases sharply around $\alpha = \alpha_c$ rapidly approaching 1, which indicates that many node pairs satisfy $\alpha \leq \alpha_c$. The sharpness of $\Phi(\alpha)$ increases for larger L , and $F(\alpha)$ becomes more step like, while α_c remains in the same location. Note that α_c can also be determined at the location where $F(\alpha|a, L)$ for increasing L cross over each other. Increasing a , on the other hand, leads to increasing α_c , consistent with path lengths becoming longer under more disordered conditions. In Fig. 2(c), we focus on $F(\alpha|a, N)$ for ER networks, and observe similar features to those in the case of lattices (we display the cumulative only as the small values of path lengths in ER graphs produce large discretization fluctuations), apart from the asymptotics which appear to be slower.

Scaling of path length as a function of disorder. In order to understand the previous results, we consider the current knowledge on the problem of optimal paths, which has received considerable attention in the context of surface growth and domain walls [14, 15, 21] in the physics literature. For the purpose of clarity we briefly review these results here, starting with lattices and extending the discussion to networks.

Disordered lattices often exhibit optimal paths which are self-affine, characterized by lengths which scale linearly with ℓ_{ij} , with a constant prefactor dependent on the roughness exponent related to the disorder [21]. This limit of self-affine paths has become known as the weak disorder limit.

Another scaling regime has been recognized [15] when the disorder approaches the so-called strong disorder limit. In this limit, each link weight in the network is very different to any other link weight, progressively forcing the optimal paths to lie inside the minimum spanning tree where their lengths scale as $\ell_{ij}^{d_{\text{opt}}}$, d_{opt} being the scaling exponent of the shortest path in the minimum spanning tree [14, 16, 22].

A general theory explaining these optimal path limits points out that weak and strong disorder are separated by a disorder length-scale ξ which depends on the disorder distribution $P_a(w)$ and some lattice dependent features [14, 17]. Optimal paths covering a distance smaller than ξ are in strong disorder, and those covering a larger distance are in weak disorder (provided the system is large enough so that $\xi \ll L$). Thus, ξ is the weak-strong disorder crossover length. The weak and strong disorder scaling regimes for ℓ'_{ij} can be expressed by the scaling relation

$$\ell'_{ij} \sim \xi^{d_{\text{opt}}} f\left(\frac{\ell_{ij}}{\xi}\right), \quad f(x) = \begin{cases} x, & x \gg 1(\text{weak}) \\ \text{const.}, & x \ll 1(\text{strong}). \end{cases} \quad (3)$$

The length-scale ξ can be determined by the relation $\xi = [p_c/(w_c P_a(w_c))]^\nu$ [14], where p_c is the percolation threshold of the lattice, ν the correlation length exponent of percolation, and w_c is the solution to the equation $p_c = \int^{w_c} P_a(w) dw$, i.e., the weight for which the cumulative distribution of $P_a(w)$ is equal to p_c .

Based on the previous arguments, we can now postulate the properties of $\Phi(\alpha)$. We concentrate on the weak disorder limit because it is the only possible regime in which an LPP transition could take place [23]. In this regime, based on Eq. (3), we find that $\ell'_{ij} \sim \xi^{d_{\text{opt}}} (\ell_{ij}/\xi) = \xi^{d_{\text{opt}}-1} \ell_{ij}$, where the parenthesis corresponds to ℓ_{ij} in the scale of the crossover length ξ , and $\xi^{d_{\text{opt}}}$ to the length of the path within this crossover length (see Fig. 3 in Ref. [17]). This relation indicates that the typical α is given by $\alpha_c \sim \ell'_{ij}/\ell_{ij} = \xi^{d_{\text{opt}}-1} \ell_{ij}/\ell_{ij} = \xi^{d_{\text{opt}}-1}$. For $P_a(w)$ of Eq. (2), $\xi = (ap_c)^\nu$ calculated according to the above formalism, producing

$$\alpha_c^{(\text{Latt})} \sim (ap_c)^{\nu(d_{\text{opt}}-1)}. \quad (4)$$

In order to test this, we present in Fig. 3(a) (main) the scaled curves $\Phi(\alpha/\alpha_c)$ for various L and a , where α_c is taken from Eq. (4). The collapse is consistent with our scaling picture, and indicates that indeed there is a clear path length increase $(ap_c)^{\nu(d_{\text{opt}}-1)}$ that explains the empirical results.

Given the large fraction of node pairs for which α is close to α_c , we postulate that

$$\tau_c^{(\text{Latt})} = \alpha_c^{(\text{Latt})}(a) \sim (ap_c)^{\nu(d_{\text{opt}}-1)}, \quad (5)$$

i.e., the tolerance necessary to obtain the LPP transition is equivalent to the most probable length factor. To test this relation, we find by inspection the values of $\tau_c^{(\text{Latt})}$ as a function of a , and plot them in Fig. 3(a) (inset). The relation between $\tau_c^{(\text{Latt})}$ and a can be fit to a power law with exponent 0.297 ± 8 , which is very close to the predicted $\nu(d_{\text{opt}}-1) = 0.293$, strongly supporting Eq. (5). In addition, we find that $\sigma^*(\tau_c^{(\text{Latt})}) \sim 0.2 \pm 0.04$, independent of a . The previous results indicate that σ^* is universal. We have tested this by comparing the shape of $\sigma^*(\tau, a, L)$ for different a , τ , and L , and have found that adjusting for a given combination of these values, the curves for σ^* can be made to overlap, supporting universality.

For ER networks, it is known that their path length distributions are concentrated due to their random structure, leading to a large number of lengths being similar to an overall typical length [18]. Thus, we simplify our analysis by focusing on the typical lengths before

and after the introduction of disorder. Before disorder sets in, the typical path length is

$$\ell_{ER} \sim \frac{\log N}{\log \langle k \rangle}. \quad (6)$$

However, after weak disorder sets in, it becomes

$$\ell'_{ER} \sim \mu a p_c \log \left(\frac{N^{1/3}}{a p_c} \right) \quad (7)$$

which emerges from the relation that is the equivalent to Eq. (3) now applied to networks. μ is a quantity not yet characterized in the literature, and depends on $\langle k \rangle$ and also weakly on N and a ; for our simulations, $\mu \approx 4$. The N -dependent ratio between ℓ_{ER} and ℓ'_{ER} , which we label $\beta(N)$ (slightly different to α because the later applies to each pair of nodes, but the former to the overall typical distances), is given by

$$\beta(N) \sim \mu a p_c \log \langle k \rangle \left[\frac{1}{3} - \frac{\log(a p_c)}{\log N} \right] = \alpha_{c,\infty}^{(ER)} - \epsilon^{(ER)}(N) \quad (8)$$

where $\alpha_{c,\infty}^{(ER)} \equiv \mu a p_c \log \langle k \rangle / 3$, the asymptotic value of β , and $\epsilon^{(ER)}(N) \equiv \mu a p_c \log \langle k \rangle \log(a p_c) / \log N$, a finite size correction that vanishes logarithmically with N as $N \rightarrow \infty$. In analogy to lattices, we hypothesize that $\tau_c^{(ER)}(N)$ scales as $\beta(N)$, which includes the size corrections with N . Also, in the limit $N \rightarrow \infty$, we define $\tau_{c,\infty}^{(ER)} \equiv \alpha_{c,\infty}^{(ER)}$.

To test Eq. (8), we introduce the rescaled variable $\alpha'_{ij} = (\alpha_{ij} + \epsilon^{(ER)}(N)) / \alpha_{c,\infty}^{(ER)}$, and plot $F(\alpha' | a, N)$ in Fig. 3(b). The collapse of the curves is excellent, and supports the validity of our assumptions. In the thermodynamic limit we expect

$$\tau_{c,\infty}^{(ER)} = \alpha_{c,\infty}^{(ER)} \sim \frac{\mu a p_c \log \langle k \rangle}{3}, \quad (9)$$

but it is important to keep in mind the large finite size corrections to $\tau_c^{(ER)}$, which make it more similar in value to Eq. (8) for finite N .

Conclusions. — We study disorder induced Limited Path Percolation and determine that a percolation transition occurs for a critical tolerance τ_c . The critical tolerance increases together with the heterogeneity of the disorder. Numerical results indicate that LPP displays universality. Also, the phase transition is first order, with the discontinuity in the order parameter $\sigma^*(\tau_c)$ independent of the disorder. For lattices $\sigma^*(\tau_c)$ is of the order of 0.2; for Erdős-Rényi networks, our numerical results suggest it may approach 1 signaling a catastrophic transition. The concept of length factor applied to the theory of optimal paths predicts a typical factor α_c which, in turn, predicts τ_c . We believe the tolerance thresholds predicted here reflect reachability conditions of some real networks under real failure scenarios (congestion, maintenance, etc.) better than regular percolation.

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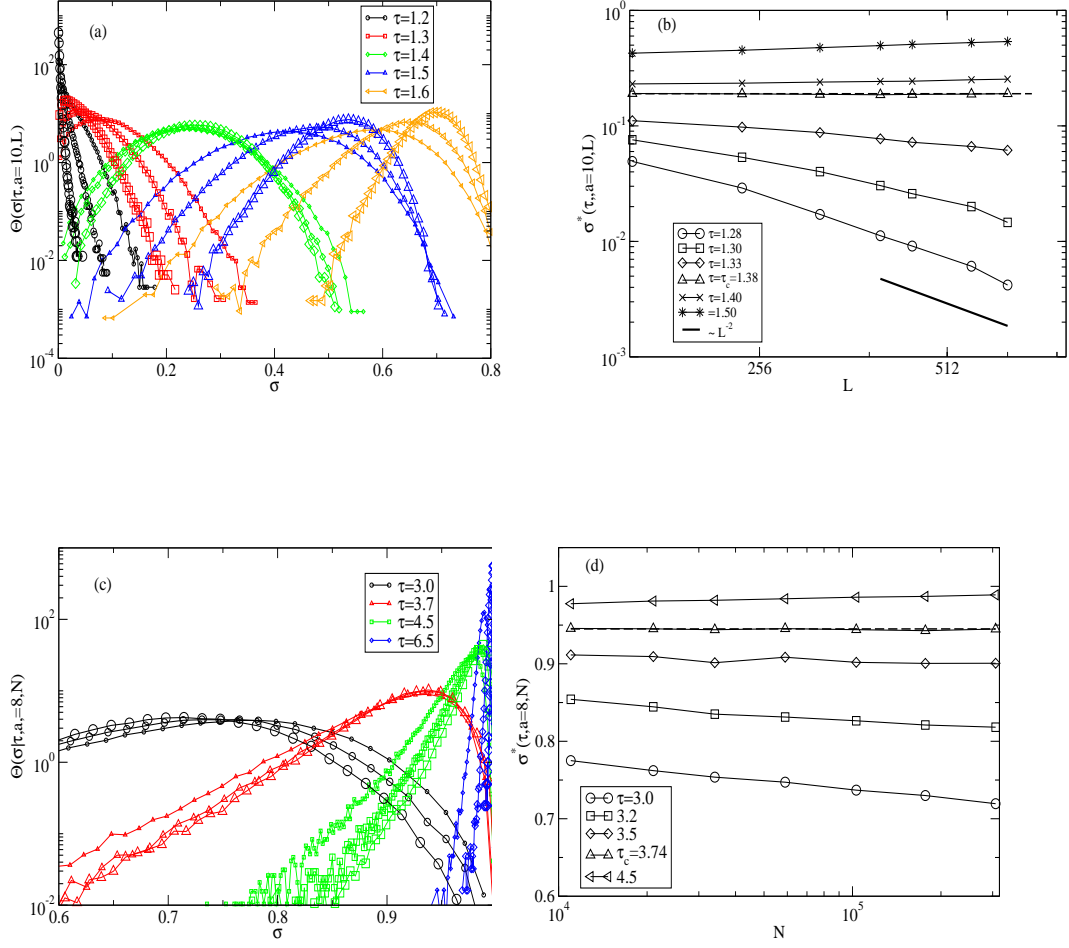


Fig. 1: (a) $\Theta(\sigma|\tau, a=10, L)$ vs. σ for the square lattice for various τ (in legend) and $L = 160, 320, 640$ (increasing symbol size corresponds to increasing system size). Note that for $\tau = 1.4$ the peaks of the distributions σ^* remain almost in the same location with increasing N indicating that $\tau_c^{(\text{Latt})}$ is near this value. For $\tau < \tau_c^{(\text{Latt})}$, σ^* decreases with N , and for $\tau > \tau_c^{(\text{Latt})}$, σ^* increases with N . Further detailed simulations reveal a better estimate of $\tau_c^{(\text{Latt})}$ is 1.38 for $a = 10$. From $\Theta(\sigma|\tau, a, L)$, we determine σ^* by cubic regression of the top points of the peak of Θ . (b) $\sigma^*(\tau, a=10, L)$ vs. L for square lattice, for several τ and L . With $\tau = \tau_c^{(\text{Latt})} = 1.38$ the curve stays constant indicating the phase transition; for τ approaching 1 (path lengths cannot increase much), σ^* scales as L^{-2} (solid line), indicating that the largest reachable component is logarithmic in size. (c) $\Theta(\sigma|\tau, a=8, N)$ vs. σ for Erdős-Rényi networks of $\langle k \rangle = 3$, $N = 11000, 59000, 307000$ (increasing symbol size corresponds to increasing system size), and various τ . In this case, for $\tau \approx 3.7$, σ^* stays roughly constant with increasing N , indicating $\tau_c^{(\text{ER})}$ is close to this value. A more detailed analysis reveals that $\tau_c^{(\text{ER})} = 3.74$ for $a = 8$ and $\langle k \rangle = 3$. (d) $\sigma^*(\tau, a=8, N)$ for ER networks, with τ values indicated in the legend, and N ranging from 11000 to 307000. It is clear that σ^* increases with N for $\tau > \tau_c^{(\text{ER})}$ and decreases for $\tau < \tau_c^{(\text{ER})}$.

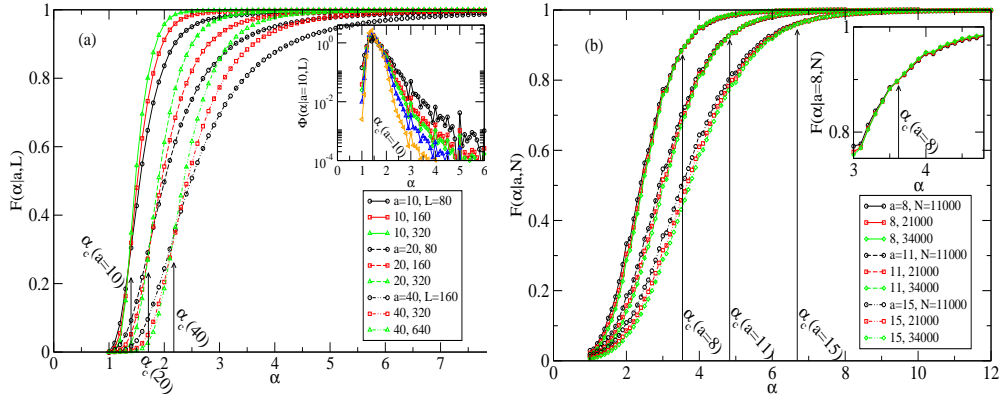


Fig. 2: Cumulative $F(\alpha|a, L)$ vs. α for 2-dimensional lattices, and $F(\alpha|a, N)$ for ER networks. Plot (a) for lattices corresponds to different L and a . The inset shows $\Phi(\alpha|a, L)$ with logarithmic scale in the vertical axis, emphasizing the fast decay of the distribution which is even more pronounced for larger system sizes. This supports the idea that, at the asymptotic limit $L \rightarrow \infty$, the distribution is highly concentrated around α_c (highlighted by the arrow). Note that the location of α_c also corresponds to the location where $F(\alpha|a, L)$ cross over each other as L increases. For increasing a , the location of α_c shifts to the right. Panel (b) for ER networks corresponds to $F(\alpha|a, N)$ for several N and a (distribution Φ is omitted due to discretization fluctuations). α_c is such that $F(\alpha|a, N)$ is approaching 1. Increasing a leads to distributions shifted to the right. The inset shows the crossing over between distributions with fixed a and increasing N .

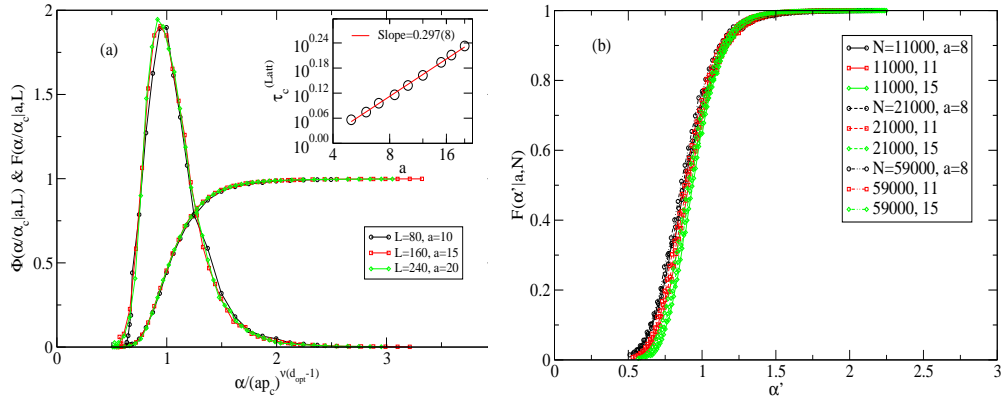


Fig. 3: Testing Eqs. (5) and (8). (a) $\Phi(\alpha/\alpha_c)$ and $F(\alpha/\alpha_c)$ vs α/α_c for lattices with $\alpha_c \sim (ap_c)^{\nu(d_{\text{opt}}-1)}$, for various a and L specified in the legend. Both Φ and F collapse supporting the hypothesis that α_c responds to Eq. (4) for lattices. Inset: $\tau_c^{(\text{Latt})}$ vs. a from data and least squares fit. The relation predicted by our theory is $\tau_c \sim (ap_c)^{\nu(d_{\text{opt}}-1)}$ which is equal to 0.293 from independent work, within error of 0.297 ± 0.008 obtained from the fit from our simulation results. (b) $F(\alpha' | a, N)$ vs. α' for ER networks for a combination of N and a . The collapse of the curves strongly supports Eq. (8).